Numerical Range of Matrices of Special Form

by

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Let $\mathcal{M}_n$ be the algebra of all $n \times n$ complex matrices. For a matrix $A \in \mathcal{M}_n$ the set

$$\mathrm{NR}[A] = \{ x^* A x \ : \ x \in \mathbb{C}^n , \quad x^* x = 1 \} \tag{A.1}$$

is called *numerical range* or *field of values* of A, R.Horn-C.Johnson, see "Topics in Matrix Analysis", pp. 1-88.

In Chapter 1 of the thesis, the $\mathrm{NR}[A]$ is expressed as the union of the numerical ranges of matrices of dimensions $k \times k$, for $2 \le k < n$, and it is proved that

$$\mathrm{NR}[A] = \bigcup_{\xi_1,\dots,\xi_k} \mathrm{NR}(\begin{bmatrix} \xi_1^* A \xi_1 & \dots & \xi_1^* A \xi_k \\ \vdots & & \vdots \\ \xi_k^* A \xi_1 & \dots & \xi_k^* A \xi_k \end{bmatrix}),$$

where $\xi_1, \dots, \xi_k$ run over all sets of k orthonormal vectors of $\mathbb{C}^n$. In this way, each set in the union can be considered as an inner approximation or *compression* of $\mathrm{NR}[A]$.

Since $\mathrm{NR}[A]$ and $\mathrm{NR}[e^{2i\theta}\bar{A}]$ are symmetric with respect to the straight line $y = (\tan\theta)\, x$, we have proved that

$$\mathrm{Co}\{ \mathrm{NR}[A] \cup \mathrm{NR}[e^{2i\theta}\bar{A}]\} = \mathrm{NR}(\frac{1}{2} \begin{bmatrix} A + e^{2i\theta}\bar{A} & -i(A - e^{2i\theta}\bar{A}) \\ i(A - e^{2i\theta}\bar{A}) & A + e^{2i\theta}\bar{A} \end{bmatrix}) \tag{A.2}$$

where $0 \le \theta \le \pi$. Therefore, $\mathrm{NR}[A]$ is presented as the intersection of numerical ranges of $2n \times 2n$ matrices on the left side in (A.2) as the line y rotates around the origin. Moreover, for $\theta = 0$ (A.2) leads to the equality

$$\mathrm{Co}\{ \mathrm{NR}[A] \cup \mathrm{NR}[\bar{A}]\} = \mathrm{NR} \begin{bmatrix} M & N \\ -N & M \end{bmatrix}$$

where $M, N \in \mathbb{R}_{n\times n}$ are defined by $A = M + iN$, and $\mathrm{NR}[A]$ lies inside the numerical range of a real matrix.

These results can be generalized if we replace in (A.1) the euclidean inner product with the *indefinite scalar product* on $\mathbb{C}^n$, since there exists an invertible hermitian matrix S, such that $< x , y >_S = y^* S x$. The *S-numerical range* of A is defined through

$$W_S[A] = \{ \frac{< Ax, x >_S}{< x, x >_S} : \ x \in \mathbb{C}^n \quad < x, x >_S \neq 0 \} = W_S^+[A] \cup W_{-S}^+[A],$$

where

$$W_S^+[A] = \{ < Ax, x >_S \; : \; x \in \mathbb{C}^n, \;\; < x, x >_S = 1 \}.$$

We present some new properties of $W_S^+[A]$, and we show that for any indefinite hermitian matrix S,

$$\mathrm{NR}[A] \cap W_S^+[A] \neq \emptyset.$$

Moreover, if the hermitian matrix S has at least one positive eigenvalue then

$$W_{I_2 \otimes S}^+[A \oplus B] \; = \; \mathrm{Co}\{ W_S^+[A] \cup W_S^+[B] \}. \tag{A.3}$$

By (A.3) we lead to equality

$$\mathrm{Co}\{W_S^+[A] \cup W_S^+[e^{2i\theta}\bar{A}]\} = W_{I_2\otimes S}^+(\frac{1}{2}\begin{bmatrix} A + e^{2i\theta}\bar{A} & -i(A - e^{2i\theta}\bar{A}) \\ i(A - e^{2i\theta}\bar{A}) & A + e^{2i\theta}\bar{A} \end{bmatrix})$$

where $0 \leq \theta \leq \pi$. The two last equalities yield

$$W_{I_2\otimes S}^+(\begin{bmatrix} A & O \\ O & \bar{A} \end{bmatrix}) = W_{I_2\otimes S}^+(\begin{bmatrix} M & N \\ -N & M \end{bmatrix}),$$

where $\theta = 0$ and $A = M + iN$, $M, N \in \mathbb{R}_{n\times n}$.

In Chapter 2 the approximation of numerical range of normal matrix A is investigated. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ $(k \leq n)$ be eigenvalues of a normal matrix $A \in \mathcal{M}_n$ such that $\mathrm{NR}[A] = \mathrm{Co}\{\lambda_1, \ldots, \lambda_k\}$ and $x_1, x_2, \ldots, x_k$ be the corresponding orthonormal eigenvectors of A. For a given unit vector $\upsilon = \sum_{j=1}^{k} c_j x_j$, $|c_j| \neq 0$ the point $\upsilon^* A \upsilon$ belongs to $\mathrm{int}\,\mathrm{NR}[A]$. Denoting $E = span\{\upsilon\}$ as subspace of $W = span\{x_1, x_2, \ldots, x_k\}$, we consider the $n \times (k-1)$ matrix $P = [w_1 \; w_2 \; \ldots \; w_{k-1}]$ where $w_1, w_2, \ldots, w_{k-1}$ is an orthonormal basis of $E_W^\perp$. Evidently, $P^*P = I_{k-1}$ and PP^* is an orthogonal projector onto $E_W^\perp$. It is proved that

$$NR[P^*AP] \subset \overline{< \lambda_1, \lambda_2, \ldots, \lambda_k >}$$

and $\partial\mathrm{NR}[P^*AP]$ is tangent to all the edges of the polygon at the points

$$\rho_\tau = \frac{|c_{\tau+1}|^2 \lambda_\tau + |c_\tau|^2 \lambda_{\tau+1}}{|c_{\tau+1}|^2 + |c_\tau|^2} \quad (\tau = 1, \ldots, k-1) \;\; ; \;\; \rho_k = \frac{|c_1|^2 \lambda_k + |c_k|^2 \lambda_1}{|c_1|^2 + |c_k|^2}.$$

Further, we structure a matrix P_1, such that $\partial\,\mathrm{NR}[P_1^* A P_1]$ is supported by some edges of $\partial\,\mathrm{NR}[A]$.
The inverse problem, where $\mathrm{NR}[G]$ is approximated outside a polygon, is investigated further. Indeed, let

$$\hat{D} = diag(\frac{p_1+p_2+3iq_2}{2}\,,\, p_1+iq_2\,,\, p_1+iq_1\,,\, \frac{p_1+p_2+3iq_1}{2}\,,\, p_2+iq_1\,,\, p_2+iq_2\,),$$

where $H(G)\,,\,S(G)$ are the hermitian parts of $G = H(G) + i\,S(G)$, and we denote $p_1 = \lambda_{min}(H(G))\ ,\ p_2 = \lambda_{max}(H(G))\ ,\ q_1 = \lambda_{min}(S(G))\ ,\ q_2 = \lambda_{max}(S(G))$.
Then we show how the $\mathrm{NR}[G]$ is dilated to a circumscribed hexagon defined by $\hat{D}$.

In Chapter 3 we consider the matrices $A_1, A_2, \ldots, A_k \in \mathcal{M}_n$ and the *joint numerical range* defined by the set

$$\mathrm{JNR}[A_1, \ldots, A_k] = \{(\,x^*A_1x\,,\,x^*A_2x\,,\ldots,\,x^*A_kx\,)\ :\ x\in\mathbb{C}^n\ ,\ \ x^*x = 1\,\}.$$

This is also called *k-dimensional field of k matrices* and, clearly, for $k = 1$ the joint numerical range is identified with the numerical range of the matrix A_1. In the sequel, it will be denoted by $\mathrm{JNR}[A_m]_{m=1}^k$. The joint numerical range is always a compact and connected set, but it is not always convex. The convexity of the joint numerical range is known for hermitian matrices when $n = k = 2$ and $n \geq 3\,,\ k \leq 3$. Here, we refer to some new properties of $\mathrm{JNR}[A_m]_{m=1}^k$, and it is proved that for a family of linearly independent hermitian bordered matrices of the form

$$S_m = \begin{bmatrix} a_{m1} & a_{m2} & \ldots & a_{mn} \\ \bar{a}_{m2} & 0 & \ldots & 0 \\ \vdots & \vdots & \mathbf{O} & \\ \bar{a}_{mn} & 0 & & \end{bmatrix} \quad ; \quad m = 1, \ldots, k$$

for $n \geq 3$ and $3 \leq k \leq 2n-1$, $\mathrm{JNR}[S_m]_{m=1}^k$ is an hyperellipsoid in $\mathbb{R}^k$ with center $\frac{1}{2}(a_{11}\,,\,\ldots\,,\,a_{k1})$ and nonempty interior. Analogue results are formulated for special 3×3 tridiagonal matrices or $(2\mu-1)$-diagonal hermitian matrices, since such matrices are presented in Graph Theory.
In the last chapter, let $\mathbb{C}[z]$ be the algebra of polynomials in one variable z with coefficients in $\mathbb{C}$, and let

$$W(z) = \left[\frac{p_{ij}(z)}{q_{ij}(z)}\right]_{i,j=1}^{n} \tag{A.4}$$

be a $n \times n$ rational matrix function, where the elements $p_{ij}(z)$, $q_{ij}(z) \in \mathbb{C}[z]$ and $q_{ij}(z)$ are not identically zero. Denoting $m(z) =$ l.c.m.$\{q_{ij}(z) : \ i, j = 1, \ldots, n\}$ we have,

$$W(z) = m(z)^{-1} P(z), \tag{A.5}$$

where $P(z) = A_m z^m + A_{m-1} z^{m-1} + \ldots + A_1 z + A_0$ is a matrix polynomial and $\deg\{m(z)\} \geq \deg\{P(z)\}$. For $W(z)$ in (A.4), the set

$$\mathrm{NR}[W(z)] = \{z \in \mathbb{C} \backslash \sigma(m) : \ x^* W(z) x = 0, \ \text{ for some nonzero } \ x \in \mathbb{C}^n\},$$

is known as the numerical range of $W(z)$, where $\sigma(m)$ is the spectrum of $m(z)$. By (A.5) we obtain

$$\mathrm{NR}[W(z)] = \mathrm{NR}[P(z)] \backslash \sigma(m)$$

where

$$\mathrm{NR}[P(z)] = \{ z \in \mathbb{C} \ : \ x^* P(z) x = 0 , \quad \text{for some nonzero} \quad x \in \mathbb{C}^n \ \}.$$

The bounds of $\mathrm{NR}[P(\lambda)]$ are known, and thus we obtain a location for the rational matrix function. Furthermore, denoting $\sigma(W) = \{z : \ \det W(z) = 0\}$ the spectrum of $W(z)$, and for $z_0 \in \sigma(W)$, there exists a nonzero vector $x_0 \in \mathbb{C}^n$, such that $W(z_0) x_0 = 0$. Hence, $z_0 \in \mathrm{NR}[W(z)]$, i.e. $\sigma(W) \subset \mathrm{NR}[W(z)]$. Moreover, $\mathrm{NR}[W(z)]$ is not always closed. Finally, a location of the derivative of the numerical range of a rational matrix function is investigated, and we see that if the roots of $m(z)$ are interior points of the ring $\Delta_2(0 \ : \ r_1, R_1)$, and $\mathrm{NR}[P(\lambda)]$ belongs to the ring $\Delta_1(0 \ : \ r, R)$, then $\mathrm{NR}[W'(z)]$ lies in the ring

$$D_1 = \{z : \min\{(r_1 , r - R_1\} \leq |z| \leq \frac{n_2 R + n_1 R_1}{n_2 - n_1}\}, \quad \text{when} \quad r > R_1,$$

or it is subset of the ring

$$D_2 = \{z \ : \ \min\{ r , r_1 - R\} \leq |z| \leq \max\{ R_1 , \frac{n_2 R + n_1 R_1}{n_2 - n_1} \}\},$$

when $R < r_1$. Then, these results are applied on the connectedness of $\mathrm{NR}[W(z)]$.

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